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On the filling of a rotating cylinder with a mixture

By M. UNGARISH AND H. P. GREENSPAN
Department of Mathematics, Massachusetts Institute of Technology,
Cambridge, Massachusetts 02139

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A rotating tank is filled from the outside with a mixture of particles and fluid. Under certain attainable conditions on the times for filling, separation, and spin-up, theory implies that the filling process acts like a centripetal separator in which heavy particles are actually concentrated at the inward-moving front. Centrifugal settling in the interior is counteracted by mass transport in the rotating boundary layers to produce this unusual volume-fraction distribution.

1. Introduction

The problem considered here is that of a rotating cylindrical container which is slowly filled with a dilute mixture of small particles (or droplets) dispersed in an incompressible fluid. The flow enters the vessel radially through the sidewall; since the filling time is considerably longer than the spin-up time the motion is essentially linear and dominated by shear layers. The corresponding motion of a homogeneous fluid in this configuration was recently examined by Ungarish & Greenspan (1984) (hereinafter referred to as U&G). In the present problem, there is the additional physical effect of centrifugal buoyancy, which forces the heavier constituent towards the periphery and causes the separation of the phases.

The primary objective is the determination of the particulate volume fraction α when the density of the particles differs only slightly from that of the fluid. For definiteness, and unless otherwise stated, the particles are taken to be heavier than the fluid. The analysis is based on a mixture model (see Ishii 1975) but before proceeding to the development of theory, it is worthwhile to describe the general and somewhat novel features of the flow.

The most interesting circumstance is when the separation and filling times are both of comparable magnitude and much longer than that for spin-up. The fluid flow is then essentially a quasi-steady, annular source–sink in a rotating cylinder, figures 1 and 2. The mixture enters through the outer wall, passes into an $E^{\frac{1}{4}}$ vertical shear layer where the flow direction is mainly axial, and then moves towards the Ekman layers on the top and bottom plates. Subsequently, the fluid is transported radially inwards by non-divergent Ekman layers, and then spreads onto the moving front (the fluid–air interface) via a vertical $E^{\frac{1}{3}}$ shear layer embedded in a weak $E^{\frac{1}{4}}$ layer. The flow in the main inviscid-fluid core is that of a potential vortex in the rotating coordinate system with zero axial and radial velocity components. However, the motion of the particles in the mixture is affected by the centrifugal buoyancy force. A heavy particle in the inviscid interior will settle toward the outer wall, but in certain circumstances it cannot completely penetrate the sidewall boundary layers. There, in the strong vertical shear flow, the particle can be sucked back into an Ekman layer, combined with newly entered mixture and transported once again to the frontal

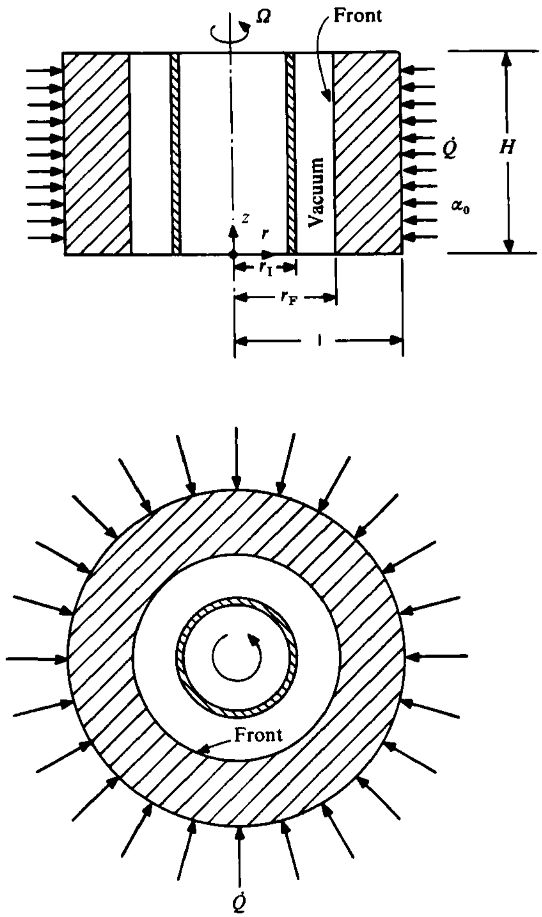

FIGURE 1. Schematic sections of the rotating container, filling through the outer wall with a mixture of volume fraction α_0.

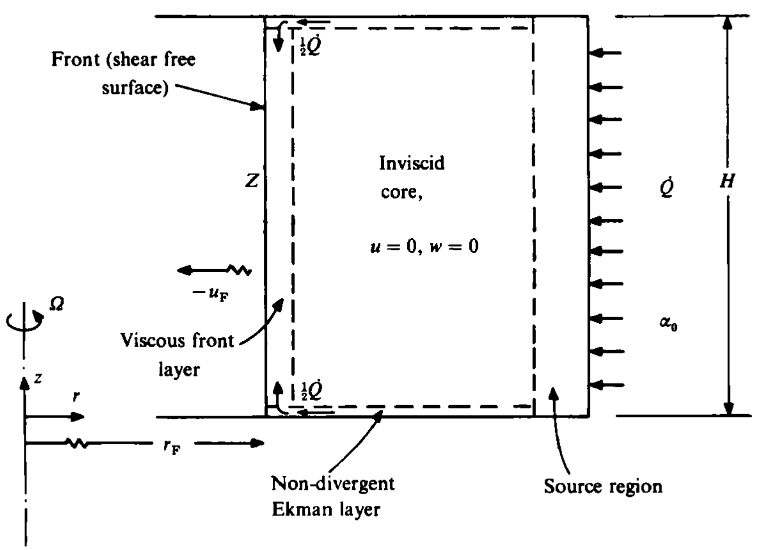

FIGURE 2. Description of the flow regions.

region. In this way, separation in the interior can actually be counteracted by the flux in the boundary layers to produce a volume-fraction distribution in which the heavy particles are more concentrated at the centre of the bowl than at the periphery. Thus the effect of centrifugal buoyancy can be successfully opposed during filling, which is a non-intuitive, perhaps surprising, prediction that provides a novel and stringent test of mixture theory.

2. Formulation

An annulus of height H^*, inner and outer radii r_I^*, r_O^*, rotates with angular velocity Ω^* about its symmetry axis, figure 1 (where an asterisk denotes dimensional variables). For simplicity, it is assumed that the tank is empty initially; the more general case in which there is at the start a layer of mixture at the outer wall can be treated similarly. At time zero, a dilute mixture with a volume fraction α_0 of dispersed particles, each of radius a^*, is made to flow slowly into the container through the sidewall at a constant, uniform volume rate $-\dot{Q}^*$, with the azimuthal velocity of the boundary.

It is convenient to scale the length by r_O^*, velocity by $V^* = \dot{Q}^*/(2\pi E^{\frac{1}{2}} r_O^{*2})$, density by ρ_C^* and time by $[(E^{\frac{1}{2}}/H)\, V^*/r_O^*]^{-1}$, where $E = \mu_C^*/\rho_C^*\, \Omega^* r_O^{*2}$ is the Ekman number and μ_C^* the viscosity of the fluid phase. (Subscripts C and D denote variables of the continuous and dispersed phases.) The dimensionless azimuthal velocity and the filling time are then of order unity; the dimensionless radial velocity of the entering fluid is $U^*/V^* = \dot{Q}^*/(2\pi r_O^* H^* V^*) = E^{\frac{1}{2}}/H$, and that of the front is also of the same order of magnitude. In dimensionless form $\dot{Q} = 2\pi E^{\frac{1}{2}}$, which is a suitably small flux. The representative Rossby number is

$$Ro = \frac{V^*}{\Omega^* r_O^*} = \frac{\dot{Q}^*}{2\pi E^{\frac{1}{2}} r_O^{*3}\, \Omega^*}. \tag{2.1}$$

In a rotating coordinate system, the dimensionless equations of motion for the mass-averaged variables of the mixture model, Ishii (1975), Ungarish & Greenspan (1984), are as follows:

mixture volume conservation:

$$\nabla \cdot \boldsymbol{q} = \nabla \cdot \left(\epsilon\alpha \frac{1-\alpha}{1+\epsilon\alpha} \boldsymbol{q}_R \right), \tag{2.2}$$

mixture momentum conservation:

$$(1+\epsilon\alpha)\left\{ 2\tilde{\boldsymbol{k}} \times \boldsymbol{q} + Ro \left[\frac{E^{\frac{1}{2}}}{H} \frac{\partial \boldsymbol{q}}{\partial t} + \tfrac{1}{2}\nabla(\boldsymbol{q}\cdot\boldsymbol{q}) + (\nabla \times \boldsymbol{q}) \times \boldsymbol{q} \right] \right\}$$
$$= -\nabla P + \frac{\epsilon\alpha}{Ro} r\hat{\boldsymbol{r}} + E(\tfrac{4}{3}\nabla(\nabla\cdot\boldsymbol{q}) - \nabla \times (\nabla \times \boldsymbol{q})] - Ro\nabla\cdot\left[\alpha(1-\alpha)\left(\frac{1+\epsilon}{1+\epsilon\alpha}\right) \boldsymbol{q}_R \boldsymbol{q}_R \right], \tag{2.3}$$

volume conservation of dispersed phase:

$$\frac{E^{\frac{1}{2}}}{H}\frac{\partial \alpha}{\partial t} + \left(\boldsymbol{q} + \frac{1-2\alpha+\epsilon\alpha^2}{1+\epsilon\alpha} \boldsymbol{q}_R \right)\cdot \nabla\alpha = -\alpha(1-\alpha)\, \nabla\cdot\boldsymbol{q}_R. \tag{2.4}$$

Here
$$\epsilon = \frac{\rho_D^*}{\rho_C^*} - 1,$$

$$\boldsymbol{q}_R = \boldsymbol{q}_D - \boldsymbol{q}_C$$

is the relative velocity, and P the reduced pressure (which includes that part of the centrifugal force that is expressible as a gradient). The second term on the right of the momentum equation represents the effective buoyancy force on the heavy dispersed particles. The next term models the averaged viscous stress, assumed here to be that of a Newtonian fluid of viscosity μ_C^*. The last term arises from the 'diffusion' of momentum induced by the relative motion between the phases.

The system of governing equations is closed here by a simple constitutive law

$$\boldsymbol{q}_R^* = \epsilon\beta\Omega^* r^* \hat{\boldsymbol{r}}, \tag{2.5}$$

where

$$\beta = \frac{2}{9}\frac{a^{*2}\Omega^*\rho_C^*}{\mu_C^*}$$

is the (modified) Taylor number of the dispersed particle. This expression for the relative velocity, which is Stokes law for settling in a centrifugal force field, can be justified for small values of β, Ro and α, to which the subsequent analysis is confined. It is emphasized, however, that the use of (2.5) in the *entire* flow field is a critical step of the present analysis. Inertial effects (Saffman 1965) and the influence of the wall affect the force balance expressed by this equation in certain regions – but to an as yet unknown extent. In this respect, the comparison of theory with experiment will be informative.

Since the dimensionless radial velocity of the entering fluid is $O(E^{\frac{1}{2}}/H)$, the convenient dimensionless form of (2.5) is

$$\boldsymbol{q}_R = \kappa\frac{E^{\frac{1}{2}}}{H} r\hat{\boldsymbol{r}}, \tag{2.6}$$

where

$$\kappa = \epsilon\beta H/(E^{\frac{1}{2}}Ro).$$

Note that κ is the ratio of the filling time to that for separation. In fact, for $\kappa > 1$, when filling is a slow process compared with settling, the particles are separated at once upon entering the container; for $\kappa \ll 1$ the container is filled long before significant separation takes place. Subsequent analysis treats only the interesting case $\kappa < 1$.

For later use, note that, with the foregoing approximations, the kinematic relationship

$$\boldsymbol{q}_D = \boldsymbol{q} + (1-\alpha)(1+\epsilon\alpha)^{-1}\boldsymbol{q}_R,$$

becomes

$$\boldsymbol{q}_D \approx \boldsymbol{q} + \kappa\frac{E^{\frac{1}{2}}}{H} r\hat{\boldsymbol{r}}. \tag{2.7}$$

The solution of the complete system of equations is, of course, a formidable task and further simplifications are necessary for analytical progress. As previously indicated, Ro and E are assumed small, which allows the elimination of the nonlinear terms in the momentum equation and the employment of boundary-layer techniques. Moreover, let $\epsilon\alpha \ll Ro$, an assumption discussed further below, in which case the equations for the mixture reduce to

$$\nabla\cdot\boldsymbol{q} = 0, \tag{2.8}$$

$$2\hat{\boldsymbol{k}}\times\boldsymbol{q} = -\nabla P - E\nabla\times(\nabla\times\boldsymbol{q}). \tag{2.9}$$

These are identical to the equations of the linear theory for homogeneous rotating fluids. Consequently, in this limit, the equations for the motion of the mixture decouple from that governing the volume fraction. Moreover, to this order of approximation, the velocity $\boldsymbol{q}$ of the mixture is the same as the single-phase flow solution discussed in U&G. (Although the linear solution is strictly valid for $Ro \ll E^{\frac{1}{4}}$, it accurately describes the flow field beyond this range.) The task now is to calculate the volume fraction α.

The important mechanisms are the *filling* throughput and the *buoyancy* separation of the mixture. The typical azimuthal velocity perturbations from solid-body rotation induced by each of these processes, are $V_F^* = \dot{Q}^*/2\pi E^{\frac{1}{2}} r_O^{*2}$ and $V_B^* = \epsilon\alpha_0 \Omega^* r_O^*$ respectively. Since subsequent analysis concerns flows in which the former effect is dominant, $V_B^* \ll V_F^*$; this implies $\epsilon\alpha_0 \ll Ro$ and gives a physical interpretation of the assumed restriction. This should limit the validity of the theory to very small values of α, i.e. to dilute suspensions, but, since in most cases of interest ϵ itself is small, the restriction on α is actually not so severe.

The substitution of (2.6) for $\boldsymbol{q}_R$ in (2.4) gives, for small α,

$$\frac{\partial \alpha}{\partial t} + (HE^{-\frac{1}{2}}u + \kappa r)\frac{\partial \alpha}{\partial r} + HE^{-\frac{1}{2}}w\frac{\partial \alpha}{\partial z} = -2\kappa\alpha, \tag{2.10}$$

where u and w are the radial and axial components of $\boldsymbol{q}$. This equation must now be solved in the various regions of the flow field.

3. Flow regimes

In the *inviscid core* $\boldsymbol{q} = -r^{-1}\boldsymbol{\theta}$, u and w are both zero (with an error $O(E^{\frac{1}{2}})$). Equation (2.10) then reduces to

$$\frac{\partial \alpha}{\partial t} + \kappa r\frac{\partial \alpha}{\partial r} = -2\kappa\alpha, \tag{3.1}$$

whose solution $\alpha = \bar{\alpha}(r, t)$ is

$$\bar{\alpha}(r, t)\, r^2 = \alpha(\gamma)\, r^2(\gamma) \tag{3.2}$$

on

$$r = r(\gamma)\exp[\kappa(t-\gamma)], \quad z = z(\gamma). \tag{3.3}$$

Here $\alpha(\gamma)$ is the volume fraction prescribed at point $r(\gamma), z(\gamma)$ at time $t = \gamma$. The radial and axial velocities in the inviscid core of the dispersed phase are, by (2.7),

$$\bar{u}_D(r, t) = \frac{E^{\frac{1}{2}}}{H}\kappa r; \quad \bar{w}_D(r, t) = 0, \tag{3.4}$$

so that (3.3) also describes the trajectory of a dispersed particle with the appropriate initial position.

Within the $E^{\frac{1}{4}}$ *source layer* at $r = 1$, the radial and axial components of $\boldsymbol{q}$ are

$$u = -\frac{E^{\frac{1}{2}}}{H}e^{-\xi}, \tag{3.5}$$

$$w = \left(\frac{2}{H}\right)^{\frac{1}{2}} E^{\frac{1}{4}} e^{-\xi}\left(\frac{z}{H} - \frac{1}{2}\right), \tag{3.6}$$

where

$$\xi = \frac{1-r}{(\frac{1}{2}HE^{\frac{1}{2}})^{\frac{1}{2}}}. \tag{3.7}$$

The velocity of the dispersed phase is then, by (2.7),

$$u_D = \frac{E^{\frac{1}{2}}}{H}(\kappa - e^{-\xi}), \tag{3.8}$$

$$w_D = w. \tag{3.9}$$

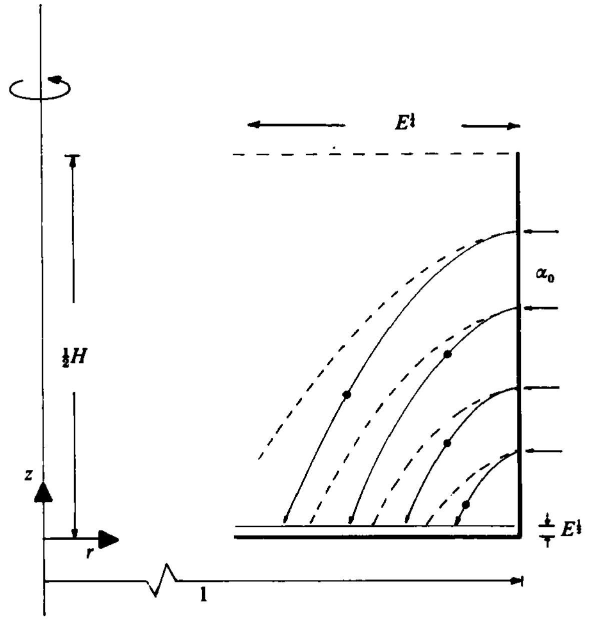

FIGURE 3. The meridional motion in the $E^{\frac{1}{4}}$ source region: ---, mixture streamlines; ——, partial paths.

Equation (3.8) implies that, for $\kappa < 1$, $u_D = 0$ in the $E^{\frac{1}{4}}$ layer at the position

$$r = r_1 = 1 + (\tfrac{1}{2}HE^{\frac{1}{2}})^{\frac{1}{2}} \ln \kappa.$$

Thus, the particles entering with the mixture at $r = 1$ clearly move directly on into the Ekman layers in the interval $r > r_1$. The dispersed particles, which are driven from the inviscid core towards the periphery by the buoyancy force, cannot reach the outer wall. At $r = r_1$, these particles acquire a considerable axial velocity (cf. (3.9) and (3.6)) and are sucked into the endwall Ekman layers, a process that radically affects the volume fraction α in the interior. The trajectories of the mixture and dispersed-phase particles are sketched in figures 3 and 4.

The substitution of (3.5) and (3.6) into (2.10) yields:

$$\frac{\partial \alpha}{\partial t} + (-e^{-\xi} + \kappa r)\frac{\partial \alpha}{\partial r} + (2H)^{\frac{1}{2}} E^{-\frac{1}{4}} \left(\frac{z}{H} - \frac{1}{2}\right) e^{-\xi} \frac{\partial \alpha}{\partial z} = -2\kappa\alpha. \tag{3.10}$$

For small E, the dominance of the last term on the left-hand side implies that $\alpha =$ const. along all pathlines in the source layer. In particular for $r_1 < r \leqslant 1$, the Ekman layers are fed by entering mixture at the imposed volume fraction α_0. However, the Ekman layers supporting the $E^{\frac{1}{4}}$ layer in the region $r < r_1$ receive particles arriving from the inviscid core. The two streams are now assumed to combine in a *mixing region* near each endwall, $1 - r = O(E^{\frac{1}{4}})$, as sketched in figure 4. We anticipate that the mixing in this region of the Ekman layers will control the distribution of the volume fraction elsewhere. The merging of the $E^{\frac{1}{4}}$ and $E^{\frac{1}{2}}$ layers at the outer wall produces a very complicated flow in which there are closed regions of recirculation as shown in figure 5. Regions of closed circulation might be expected to have a lower or even zero particle concentration, but this would imply a stratification of the effective viscosity, which is unstable in parallel flows, Yih (1967). Moreover, the buoyancy force would also contribute to the destabilization of the

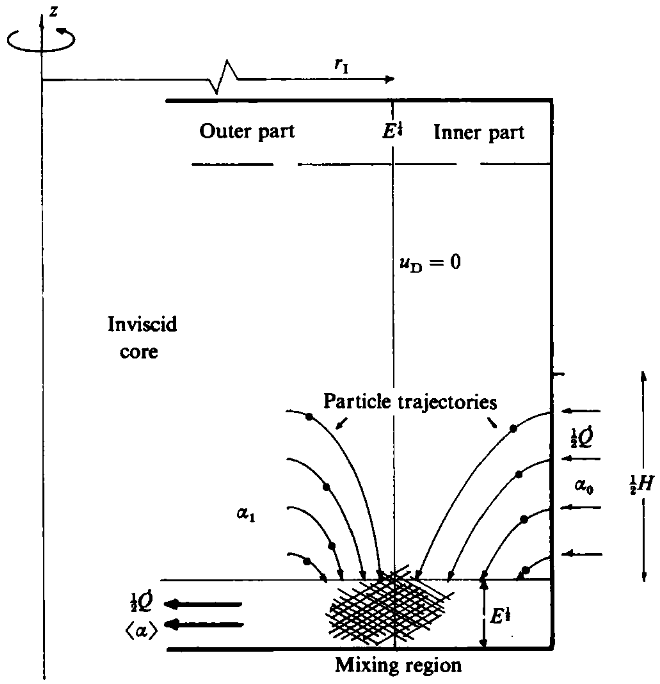

FIGURE 4. The mixing region (at bottom Ekman layer).

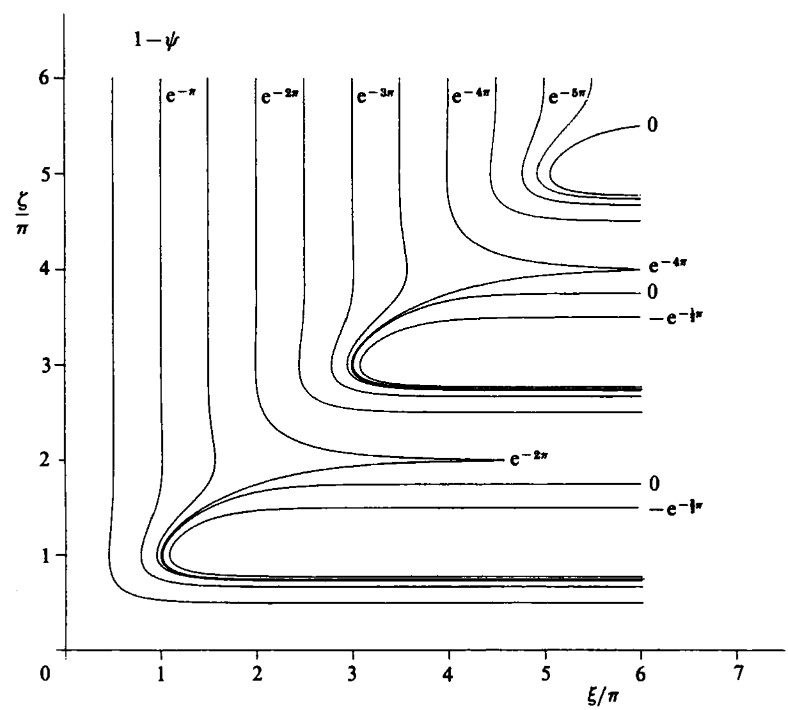

FIGURE 5. Streamlines in the $E^{\frac12}\times E^{\frac14}$ corner of a linear homogeneous fluid source at $r = 1$. The values on the lines are $(1-\psi)$, where $\psi = -E^{-\frac12} r \int_0^z u \, dz$, and the volumetric rate transported by *one* Ekman layer is $2\pi E^{\frac12}$.

delicate flow field in merging, rotational boundary layers. For these reasons some sort of flow instability must be anticipated that results in the instantaneous mixing of the fluid in the Ekman layers around $r = r_1$. The mixture that leaves this region is therefore assumed to have an homogenized volume fraction $\langle\alpha\rangle$. The value of $\langle\alpha\rangle$ is obtained from the conservation law for particle volume. The volume flux of particles arriving from the inner core applied to the annulus at $r = r_1 \approx 1$ is, by (3.4),

$$\dot{Q}_{D_1} = 2\pi r_1 H\bar{u}_D(1)\,\bar{\alpha}(1,t) \approx 2\pi E^{\frac{1}{2}}\kappa\bar{\alpha}(1,t), \tag{3.11}$$

where the additional assumption is made (to be rationalized later) that $\bar{\alpha}$ does not depend on z. Therefore

$$\dot{Q}_{D_1} + \dot{Q}\alpha_0 = (\dot{Q} + \dot{Q}_{D_1})\langle\alpha(t)\rangle \tag{3.12}$$

and since in the dilute limit $\dot{Q}_{D_1} = o(\dot{Q})$ it follows that

$$\langle\alpha(t)\rangle = \alpha_0 + \kappa\bar{\alpha}(1,t). \tag{3.13}$$

The volume fraction $\langle\alpha\rangle$, which is obviously larger than the entry value α_0, is fed into the non-divergent Ekman layers bounding the inviscid core. There

$$w = 0, \quad u = -\frac{1}{r}\,\mathrm{e}^{-\zeta}\sin\zeta, \tag{3.14}$$

where ζ is the axial distance from the boundary stretched by $E^{-\frac{1}{2}}$.

Combining (3.14) and (2.10) gives

$$\frac{\partial\alpha}{\partial t} + O(E^{-\frac{1}{2}})\frac{\partial\alpha}{\partial r} = -2\kappa\alpha, \tag{3.15}$$

indicating that $\alpha = \text{const.}$ along pathlines that are essentially parallel and horizontal. In view of the mixing postulated in the feed region $r = 1$, and the possibility of other instabilities in the recirculations of the Ekman layer due to buoyancy, viscous and inertial effects, we take

$$\alpha = \langle\alpha(t)\rangle \tag{3.16}$$

everywhere in the boundary layer. In other words, the typical radial flow in the Ekman layers is very vast compared to $|\boldsymbol{q}_R|$, and the short time spent by a mixture particle in this region is insufficient for any significant separation to have occurred.

The *frontal region* consists of a moving $E^{\frac{1}{3}}$ shear layer at the front, $r = r_F(t)$; its location is given by

$$r_F^2 = 1 - 2t. \tag{3.17}$$

This layer receives an $O(E^{\frac{1}{2}})$ volume flux from the Ekman layer. The detailed examination of the complicated dispersed-phase motion in this region is not attempted; instead, a global analysis is adopted.

The $O(E^{\frac{1}{3}})$ axial velocity in the front layer is relatively large, so that particles should spread in a manner that produces a homogeneous volume fraction $\alpha(r_F, t)$. Local instabilities are also expected to assist in this process.

Since the particulate and fluid volume in all the thin boundary layers is small, global volume conservation of the dispersed phase implies:

$$2\pi H^* \int_{r_F^*(t^*)}^{r_{1-}^*} \bar{\alpha}(r^*, t^*)\, r^*\,\mathrm{d}r^* = \alpha_0\dot{Q}^* t^*, \tag{3.18}$$

or, in the dimensionless form,

$$\int_{r_F(t)}^{1-} \bar{\alpha}(r, t)\, r\, \mathrm{d}r = \alpha_0 t. \tag{3.19}$$

The time derivative of this equation and the use of (3.17) and (3.1) give

$$\bar{\alpha}(r_F, t)\,[1+\kappa r_F^2(t)] - \kappa\bar{\alpha}(1, t) = \alpha_0. \tag{3.20}$$

Since by (3.13) $\langle\alpha(t)\rangle = \alpha_0 + \kappa\bar{\alpha}(1, t)$, this implies

$$\frac{\bar{\alpha}(r_F, t)}{\langle\alpha(t)\rangle} = \frac{1}{1+\kappa r_F^2(t)} < 1. \tag{3.21}$$

Thus the volume fraction is much reduced when mixture enters the inviscid core from the front via the boundary layers. This effect can be attributed to the motion of the front towards the centre while particles tend to move in the opposite direction.

4. Volume-fraction distribution

The volume fraction in the core $\bar{\alpha}(r, t)$, can be obtained by appropriately matching the solutions in the different flow regions discussed in the last section. A straightforward discrete method of computation is to follow the motion of a particle in a container which initially has a thin layer of fluid at the outer wall $b \leqslant r \leqslant 1$. The particle at $r = b$ reaches $r = r_1 = 1$ in time $\Delta t = (\ln b)/\kappa$ (cf. (3.3)) from where, in a negligible time interval, it is transported to the front whose position is now $r_F^2(\Delta t) = b^2 - 2\Delta t$. The journey towards the periphery is repeated, and so on. The value of α on any point of this trajectory (and, in particular, at r_1 and r_F) can be calculated via (3.2), (3.13), (3.16) and (3.20). The case of an initially empty container is recovered in the limit $b \to 1$.

The analytical approach is to solve (3.1) in the domain $r_F(t) \leqslant r \leqslant 1$ subject to the constraint (3.20). (The container is taken to be initially empty, $r_F(0) = 1$.) Note that this domain does not include the shear layers.

The general solution of (3.1) is

$$\bar{\alpha}(r, t) = \frac{\alpha_0}{r^2} G(r^2\, \mathrm{e}^{-2\kappa t}), \tag{4.1}$$

where G is an arbitrary function that must satisfy the above mentioned constraint. This yields the functional equation for G:

$$G(\eta\, \mathrm{e}^{\kappa(\eta-1)})\,(1+\kappa\eta) - \kappa\eta G(\mathrm{e}^{\kappa(\eta-1)}) = \eta, \tag{4.2}$$

where $\eta = r_F^2 = 1 - 2t$.

Equation (4.2) can be solved by several methods. Numerical solutions are displayed in figure 6. A Taylor-series expansion in powers of $(1-\eta)$ yields, for $\kappa \neq -\frac{1}{2}$,

$$G(\eta) = 1 - a_2(1-\eta) - \tfrac{1}{2}\kappa a_2 a_3(4+3\kappa)\,(1-\eta)^2 - \tfrac{1}{6}\kappa^2 a_2 a_4[3a_3(1+\kappa)\,(1+3\kappa)^2 - (9+4\kappa)]\,(1-\eta)^3 - \ldots, \tag{4.3}$$

where $$a_n = [(1+\kappa)^n - \kappa^n]^{-1}.$$

The number of terms that must be used may be gauged from the strict condition $G(0) = 0$.

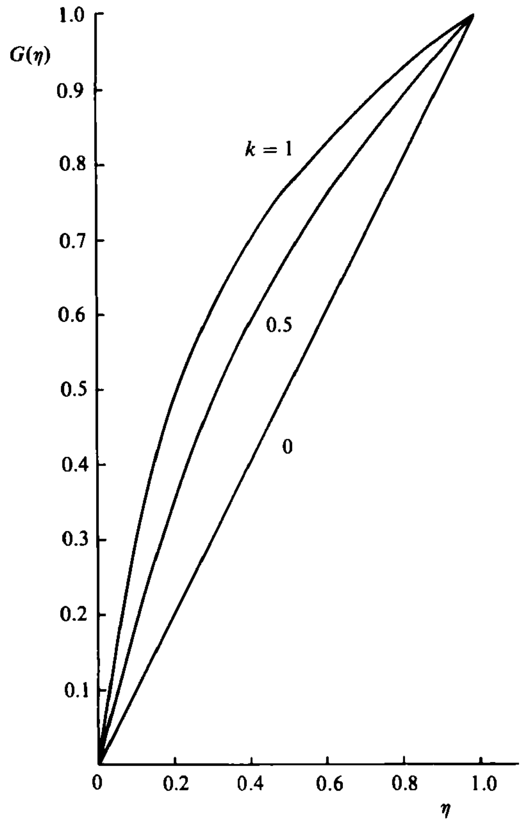

FIGURE 6. The function $G(\eta)$, numerical calculation via a recursion formula.

A perturbation series valid for small κ gives similar results and a simple formula for the volume fraction is

$$\alpha(r,t)/\alpha_0 \simeq 1+2\kappa(1-r^2-t). \tag{4.4}$$

The most accurate results approximated for small κ are summarized as follows:

in the core $$\bar{\alpha} = \frac{\alpha_0}{r^2} G(r^2(1-2\kappa t)),$$

so that by (4.3) $$\bar{\alpha} = \alpha_0 \left(\frac{1-2\kappa t}{1+2\kappa}\right) \quad \text{as } r \rightarrow 1^-; \tag{4.5a}$$

and from (3.20) $$\bar{\alpha} = \alpha_0 \left(\frac{1+2\kappa t}{1+\kappa}\right) \quad \text{as } r \rightarrow r_{\mathrm{F}^+}; \tag{4.5b}$$

in the Ekman layer, (3.13) implies

$$a = \langle \alpha \rangle = \alpha_0 \left(1+\kappa - \frac{2\kappa^2 t}{1+\kappa}\right); \tag{4.6}$$

in the sidewall layer (3.16) yields

$$\alpha = \alpha_0 \quad \text{for } r > r_1. \tag{4.7}$$

The accuracy of (4.5a, b) has been verified by comparison with numerical solutions, figure 7.

The most striking result is that the volume fraction is largest at the front and decreases towards the outer wall, to a value $\alpha(1,t)$ that is *lower* than that of the

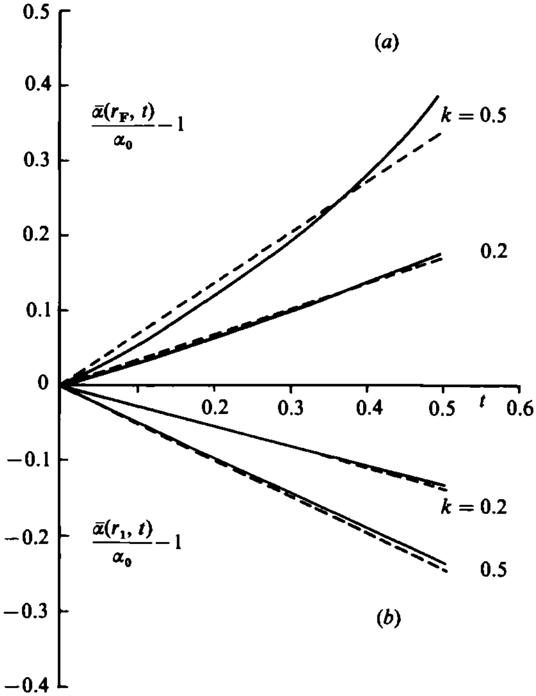

FIGURE 7. Behaviour of the reduced volume fraction $[(\alpha/\alpha_0)-1]$ *vs* time at: (*a*), $r = r_F(t)$; (*b*), $r = r_1 \approx 1$. ——, numerical solution; ---, approximations (4.5*a*) and (4.5*b*).

entering fluid. As filling proceeds, the concentration of particles at the front as well as the deficit at the wall both increase, a transient state that is in complete opposition to the centrifugal buoyancy force.

If the particles are lighter than the fluid, an analogous situation develops in which the largest particle concentration is at the entry wall while the lowest is at the moving front. In this case, (4.4) holds with $-\kappa$ substituted for κ at least for small κ. However, the explanation of this concentration profile is slightly different because there is now no important mixing zone in the sidewall boundary layer. The radial velocity of particles entering the tank remains positive in the vertical shear layer, and most of them proceed on directly into the interior. Fewer particles enter the Ekman layer, which then transports a mixture with a low concentration of dispersed matter to the front. Light particles pile up at the entry wall because of the small settling velocity while purer fluid is moved inwards by the Ekman layers to fill the container.

The preceding analysis is valid during the filling process, $t < t_F$, where $t_F = \frac{1}{2}(1-r_1^2)$. Once filling is completed, a spin-down process takes place in which columns of a mixture undergo an $O(Ro)$ radially outward displacement. The volume fraction in the interior, however, remains unaffected since the separation time is large compared to the spin-up interval. This stage is followed by separation of the mixture, which is essentially rotating rigidly. The governing equation is

$$\frac{\partial \alpha}{\partial t} + \kappa r \frac{\partial \alpha}{\partial r} = -2\kappa\alpha, \tag{4.8}$$

subject to the initial condition $\alpha_*(r) = \alpha(r, t_F)$ obtained from the filling solution. It follows that: $\alpha = \alpha_*(r_*)\,e^{-2\kappa t}$ on $r = r_*\,e^{\kappa t}$ for $t > t_F$. A sediment layer develops on the outer wall and a kinematic shock, whose locus is $r = r_I\,e^{\kappa t}$, separates between the purified fluid in the central region and the separating bulk of mixture. Except for the variable initial volume fraction, the radial flow is much like that discussed by Greenspan (1983).

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